Kinetic Equations

Text of the Exercises

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Teachers: Prof. Chiara Saffirio, Dr. Théophile Dolmaire
Assistant: Dr. Daniele Dimonte – daniele.dimonte@unibas.ch

Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T]\times\mathbb{R}^d\right)$ be bounded with $\operatorname{div}_x b$ bounded. Assume that $u_0 \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ and $f \in L^1\left([0,T]\,;L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right)$.

Prove that there exists a unique function $u \in L^\infty\left([0,T]\,;L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right)$ such that for any $\varphi \in C^\infty_{\mathrm{c}}\left([0,T]\times\mathbb{R}^d\right)$ the map $t \mapsto \langle u(t,\cdot),\varphi\rangle$ is continuous in t and which is solution to

$$\begin{cases} \partial_t u + b\cdot\nabla_x u = f, & \text{in } \mathcal{D}'\left([0,T]\times\mathbb{R}^d\right), \\ u|_{t=0} = u_0, & \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \end{cases} \tag{1}$$

Exercise 2

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T]\times\mathbb{R}^d\right)$ be bounded with $\operatorname{div}_x b$ bounded. Assume that $u_0 \in C^1\left(\mathbb{R}^d\right)\cap L^\infty\left(\mathbb{R}^d\right)$. Thanks to the first exercise, we now that there exists $u \in L^\infty\left([0,T]\,;L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right)$ such that for any $\varphi \in C^\infty_{\mathrm{c}}\left([0,T]\times\mathbb{R}^d\right)$ the map $t \mapsto \langle u(t,\cdot),\varphi\rangle$ is continuous in t and which is a solution to

$$\begin{cases} \partial_t u + b\cdot\nabla_x u = 0, & \text{in } \mathcal{D}'\left([0,T]\times\mathbb{R}^d\right), \\ u|_{t=0} = u_0, & \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \end{cases} \tag{2}$$

Prove that the following statements are equivalent:

- $u \in L^\infty\left([0,T]\times\mathbb{R}^d\right)$ is a renormalized solution to (2);
- $u \in C^1\left([0,T]\times\mathbb{R}^d\right)$ is a classical solution to (2).

Exercise 3

Let Ω_1 and Ω_2 be two measurable spaces with σ-finite measures μ_1 and μ_2 respectively. Let $f:\Omega_1\times\Omega_2\to\mathbb{R}$ be a $\mu_1\times\mu_2$ measurable function and assume that $f\geqslant 0$. Let $p\in[1,+\infty)$. Then

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)\,d\mu_2(y)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leqslant \left(\int_{\Omega_1}\int_{\Omega_2} f(x,y)^p\,d\mu_2(y)\,d\mu_1(x)\right)^{\frac{1}{p}} \tag{3}$$